Integral Calculus Concepts

Area

It seems reasonable that the area under a curve $y = f(x)$ continuous over an interval (a, b) can be approximated to any degree of accuracy by vertically slicing it into a sum of rectangles as suggested by Figure 1. Divide the interval into n subintervals (x_{i-1}, x_i) by the $n + 1$ points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$,then

$$f(x_1^*)dx_1 + f(x_2^*)dx_2 + \cdots + f(x_n^*)dx_n, \quad x_{i-1} \le x_i^* \le x_i, dx_i = x_i - x_{i-1}, i = 1, \cdots, n \tag{1}$$

is such an approximate sum. x_i^* can be any point in the interval (x_{i-1}, x_i) where we evaluate $f(x)$ to get the height of a rectangle. When the limit of the sums of finer and finer subdivisions exists, we say $f(x)$ is (Riemann) integrable which is so if $f(x)$ is continuous and it is called the definite integral of $f(x)$ from a to b and is denoted by $\int_a^b f(x)dx$. The ‘S’ shaped symbol $\int_a^b$ connotes summing of the rectangular slices $f(x)dx$.

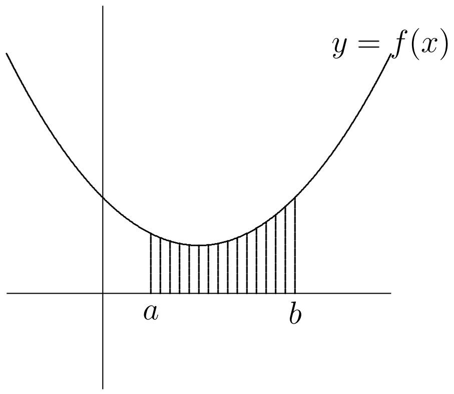

Figure 1. Area by rectangular slices

Indefinite Integral

$F(x)$ is called an indefinite integral of $f(x)$ if $F'(x) = f(x)$; that is, $F(x)$ has $f(x)$ as its derivative. Thus, the problem of indefinite integration is finding a function $F(x)$ whose derivative is $f(x)$. For example, what is the function whose derivative is x^n? It is $\dfrac{x^{n+1}}{n+1} + c$ (a constant) by taking the latter's derivative. Unfortunately, there is a host of functions, for example, e^{-x^2} is one, that cannot be integrated in terms of finite algebraic operations and compositions of the so-called elementary functions; namely, the powers of x, exponentials, trigonometric functions, and their inverses. This means integration is often difficult, even intractable, and is the reason for integral tables.

Fundamental Theorem of Calculus

If $f(x)$ is integrable over (a, b) and has indefinite integral $F(x)$, then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a).$$

Proof of the Fundamental Theorem of Calculus

We use the observation shown in Figure 2, a consequence of the mean value theorem for derivatives, that there is at least one point, say x_i^*, in the interval (x_{i-1}, x_i) whose tangent slope equals the secant slope:

$$F'(x_i^*) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}, \tag{2}$$

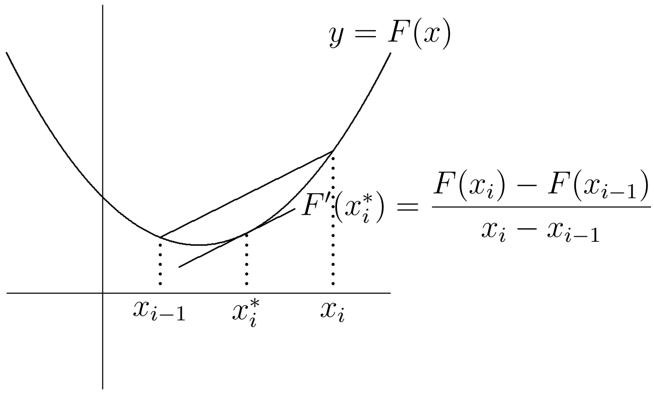

Figure 2. Tangent slope equals secant slope

The proof proceeds as follows:

$$\begin{aligned} f(x_i^*)dx_i &= F'(x_i^*)dx_i && \text{(indefinite integral)} \\ &= F'(x_i^*)(x_i - x_{i-1}) && \text{(definition of } dx_i\text{)} \\ &= F(x_i) - F(x_{i-1}) && \text{(tangent slope = secant slope),} \end{aligned}$$

this last equality being a simple rearrangement of (2). Substituting this last expression for $f(x_i^*)dx_i$ into (1), we get

$$F(x_1) - F(x_0) + F(x_2) - F(x_1) + F(x_3) - F(x_2) + \cdots + F(x_n) - F(x_{n-1})$$

that is $F(b) - F(a)$ because all terms cancel save the second and the second from the last. Since this is true for any subdivision of (a, b), it is true for the limit which exists because by assumption $f(x)$ is integrable. Q.E.D.

Example of the Fundamental Theorem of Calculus

$$\int_a^b x^n dx = \int_a^b (\frac{x^{n+1}}{n+1})' dx = \frac{x^{n+1}}{n+1}|_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}, \quad n > -1$$

Definite Integral Rules

The following properties are inherited from the properties of sums of signed rectangles whose heights $f(x_i^*)$ and bases dx_i appearing in (1) can be either $\pm$; thus, areas can be positive, negative, or zero.

$\int_a^b (cf(x)+g(x))dx$	$=$	$c\int_a^b f(x)dx + \int_a^b g(x)dx,$	*Linearity*
$\int_a^a f(x)dx$	$=$	0	*Empty interval*
$\int_b^a f(x)dx$	$=$	$-\int_a^b f(x)dx$	*Reverse integration direction*
$\int_a^b f(x)dx$	$=$	$\int_a^c f(x)dx + \int_c^b f(x)dx$	*Split interval*
$\int_a^b f(x)dx$	$\geq$	$0, \quad f(x) \geq 0$	*Non − negativity*
$\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$	$=$	$f(c), \text{some } c \text{ in } (a,b)$	*Average of $f(x)$ on (a,b)*
$\int_a^b f'(x)dx$	$=$	$f(b)-f(a)$	*Integral of derivative*
$\dfrac{d}{dx}\displaystyle\int_a^x f(t)dt$	$=$	$f(x)$	*Derivative of integral*
$\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)dt$	$=$	$f(u(x))\cdot\dfrac{du}{dx}$	*Chain rule*
$\int_a^b f(u(x))\dfrac{du}{dx}(x)dx$	$=$	$\int_{u(a)}^{u(b)} f(u)du$	*Change of variable*
$\int_a^b f(u(x))dx$	$=$	$\int_{u(a)}^{u(b)} f(u)\dfrac{dx}{du}(u)du$	*Change of variable*
$\int_a^b u(x)dv(x)$	$=$	$u(x)v(x)\vert_a^b - \int_a^b v(x)du(x)$	*Integration by parts*

Integration by parts comes from integrating the derivative of the product in the form

$$u\cdot\frac{dv}{dx} = \frac{d(u\cdot v)}{dx} - v\cdot\frac{du}{dx}.$$

As examples of applying the rules, verify:

$$\int_1^e \ln x dx = 1, \quad \frac{d}{dx}\int_1^{\sqrt{x}} t dt = \frac{1}{2}, \quad \text{and} \quad \int_0^1 \frac{dx}{(x+1)^2} = \frac{1}{2}$$

via integration by parts, chain rule, and change of variable, respectively.

Caution!

Beware of formal symbol substitution as the following example shows. Since $\frac{1}{x^2} \geq 1$ on $(-1,1)$, how can it be that

$$\int_{-1}^1 \frac{dx}{x^2} = \int_{-1}^1 (-\frac{1}{x})'dx = -\frac{1}{x}|_{-1}^1 = -1-1 = -2?$$

Graph $f(x) = \frac{1}{x^2}$ and estimate its integral by a small sum of rectangles. What is the height of a rectangle if one chooses $x_i^* = 0$?

The rule is: if the values of $f(x)$ on (a,b) become infinite, then we must evaluate the integral as a limit over where $f(x)$ is finite.